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Asymptotic Behavior of the Current on an Infinite Cylindrical Antenna

K. S. Kunz

Contribution from Research Center, New Mexico State University, University Park, New Mexico

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An asymptotic expression is obtained for the current distribution on the outside surface of an infinitely long, perfectly conducting, hollow cylindrical antenna that is fed by an infinitesimally narrow circumferential gap. This asymptotic expression involves two series. The first series is expressed in reciprocal powers of log $(2|z|/j\Gamma^2ka^2)$, where $|z|$ is the distance from the gap, log Γ is Euler's constant, k is the propagation constant, and a is the radius of the antenna. The second series is a similar series multiplied by $1/(k|z|)$. The first series is dominant and its first five terms yield values for the magnitude and phase of the current that for even moderately thick antennas (circumferences as large as $\lambda/3$) are accurate to within about one percent in as close as $\lambda/3$ of the gap. This is shown by a comparison of the values of the current obtained from these terms with the numerically computed values of Duncan [1962]. Asymptotic expressions for the current found in the literature resemble the first term of this dominant series and are accurate only at relatively large distances from the gap—except for very thin antennas.

1. Introduction

Consider the usual model of an infinite antenna composed of a hollow circular cylinder with its axis lying along the z-axis of a cylindrical coordinate system with coordinates (ρ, ϕ, z). The cylinder is cut in two at $z=0$ and the antenna is assumed to be excited by a sinusoidal generator producing an oscillating electric field $E_z e^{j\omega t}$ (that is independent of ϕ) across the gap in the cylinder formed by the cut. The voltage across the gap is

$$V = -\int_{gap} E_z dz$$

which may be held constant while the gap width is allowed to approach zero. As is well known this limiting process of allowing the gap width to go to zero leads to a singularity of the quadrature component of the current $I(z)$ on the outer surface of the cylinder. The effect of such a gap has been considered, among others, by Infeld [1947], King [1956], Wu and King [1959], Chen and Keller [1962], and Duncan [1962].

The current on the outer surface of the cylinder is given by the formula [1]

$$I(z) = \frac{kaV}{jZ_0}\int_{-\infty}^{\infty} \frac{H_1^{(2)}(\beta a)e^{-j\gamma|z|}d\gamma}{\beta H_0^{(2)}(\beta a)} \tag{1}$$

where

$k = 2\pi\lambda^{-1}$
λ = wavelength in free space
a = radius of antenna
$j = \sqrt{-1}$
$Z_0 = \sqrt{\mu/\epsilon}$
ϵ = dielectric constant of free space
μ = permeability of free space
$\beta = (k^2-\gamma^2)^{\frac{1}{2}}$
$H_0^{(2)}(\beta a) = J_0(\beta a) - jY_0(\beta a)$
$H_1^{(2)}(\beta a) = J_1(\beta a) - jY_1(\beta a).$

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[1] See formula 13 of Duncan [1962].

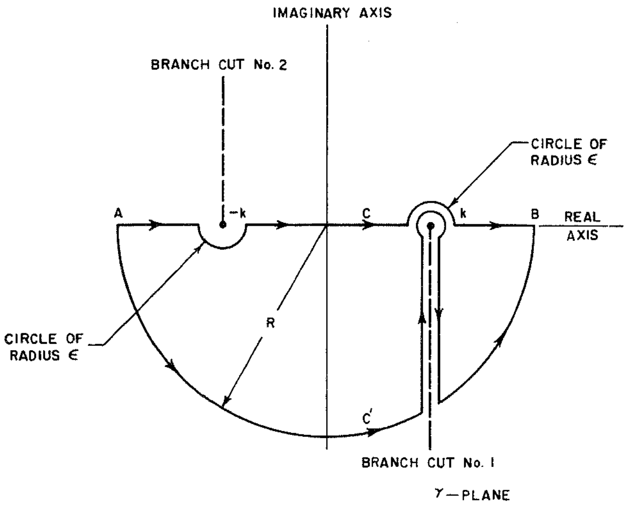

FIGURE 1. *Contours of integration in the γ-plane.*

The last two functions are Hankel functions of the second kind, the time factor having been taken as $e^{j\omega t}$. The path of integration in (1) is along C in figure 1 and is thus along the real axis except at $\gamma=-k$ where it is downwardly indented and at $\gamma=k$ where it is upwardly indented to avoid the branch points of β at $\gamma=\pm k$. The angle of β, written $\angle\beta$, is chosen so that $\angle\beta=0$ for $-k<\gamma<k$.

Duncan [1962] has recently derived expressions for $I(z)$ that permit its computation for very small values of z and has obtained accurate plots of the real and imaginary parts of $I(z)$ as one approaches the singularity at $z=0$. While this approach also makes possible accurate values of $I(z)$ out to $\zeta=k|z|$ of 5 or 10, the work involved increases and the accuracy decreases as ζ becomes large.

This paper develops an asymptotic series for the current

$$I(z)\cong\frac{2\pi V}{Z_0}e^{-j\zeta}\left[\sum_{n=1}^{(\infty)}\frac{A_n}{(\log p\zeta)^n}+\frac{j}{\zeta}\sum_{n=2}^{(\infty)}\frac{B_n}{(\log p\zeta)^n}\right] \tag{2}$$

where A_n and B_n are constants and

$$p=\frac{2}{j\Gamma^2k^2a^2}. \tag{3}$$

It is shown that the first five terms of the first series yield a value of $I(z)$ that is remarkably accurate even for rather small values of $\zeta=k|z|$ and for thick antennas. Thus, for example (see figs. 2 and 3), $|I(z)|$ is given within an accuracy of 1 percent for all antennas thinner than $\alpha=2\pi a\lambda^{-1}=0.30$ for all $\zeta=2\pi|z|\lambda^{-1}>1.8$ (approx.).

2. Modification of the Integral for $I(z)$ by Contour Integration

This integral for $I(z)$ given in (1) may, by reference to figure 1, be written

$$I(z)=\frac{kaV}{jZ_0}\lim_{\substack{\epsilon\to 0\\ R\to\infty}}\int_C\frac{H_1^{(2)}(\beta a)e^{-j\gamma|z|}}{\beta H_0^{(2)}(\beta a)}d\gamma. \tag{4}$$

By distorting the contour C between A and B into C', one sees that C in (4) can be replaced by C'. It can readily be shown that in the limit of $\epsilon\to 0$ and $R\to\infty$ only the portions of the path C' lying along branch cut 1 contribute to the integral, and hence

$$I(z)=\frac{jkaV}{Z_0}\int_k^{k-j\infty}\left[\frac{H_1^{(2)}(\beta a)}{\beta H_0^{(2)}(\beta a)}-\frac{H_1^{(2)}(\beta ae^{-j\pi})}{\beta e^{-j\pi}H_0^{(2)}(\beta ae^{-j\pi})}\right]e^{-j\gamma|z|}d\gamma. \tag{5}$$

Decreasing the angle of the argument βa of the Hankel functions by π leads to the following equations:

$$H_0^{(2)}(\beta a e^{-j\pi}) = -J_0(\beta a) - jY_0(\beta a) = -H_0^{(1)}(\beta a)$$
$$H_1^{(2)}(\beta a e^{-j\pi}) = J_1(\beta a) + jY_1(\beta a) = H_1^{(1)}(\beta a). \tag{6}$$

These together with the substitution $\gamma = k(1-jy)$ and the well-known identity

$$J_1(x)Y_0(x) - J_0(x)Y_1(x) = \frac{2}{\pi x} \tag{7}$$

makes it possible to write (5) in the form

$$I(z) = \frac{j4Ve^{-j\zeta}}{\pi Z_0} \int_0^\infty \frac{e^{-y\zeta}dy}{\eta^2[J_0^2(\eta\alpha) + Y_0^2(\eta\alpha)]} \tag{8}$$

where

$$\zeta = k|z|, \; \alpha = ka, \text{ and } \eta = \sqrt{y(y+2j)}. \tag{9}$$

The change in the contour of integration thus leaves us with a more manageable integral for large values of ζ in that only the small argument behavior of the Bessel functions is important.

3. More Tractable Expression for the Asymptotic Behavior of $I(z)$

We may write (8) in form

$$I(z) = \frac{2Ve^{-j\zeta}}{\pi Z_0} T \tag{10}$$

where

$$T = \frac{2j}{\zeta} \int_0^\infty \frac{e^{-u}du}{\eta^2[J_0^2(\eta\alpha) + Y_0^2(\eta\alpha)]} \tag{11}$$

and

$$\eta = \frac{1}{\zeta}\sqrt{u(u+2j\zeta)}. \tag{12}$$

Within the region $|\eta\alpha| < b$, where b is a small positive number, the Bessel functions may be replaced by their small argument approximations. This amounts to setting

$$\frac{1}{D(\eta\alpha)} \equiv \frac{1}{J_0^2(\eta\alpha) + Y_0^2(\eta\alpha)} = \frac{1 + \epsilon_1(\eta\alpha)}{1 + \left(\frac{2}{\pi} \log \frac{\Gamma\eta\alpha}{2}\right)^2} \tag{13}$$

and neglecting $\epsilon_1(\eta\alpha)$. This leads to a maximum relative error in the approximation that is a function of b. For $b=0.5$ this maximum error is about 1 percent.

Letting u_1 be the value of u for which

$$|\eta\alpha| = \frac{\alpha}{\zeta}\sqrt{u_1|u_1 + 2j\zeta|} = b \tag{14}$$

we have

$$u_1 = C\zeta \tag{15}$$

where

$$C = \left\{ 2\left[\left(1 + \frac{b^4}{4\alpha^4}\right)^{1/2} - 1\right]\right\}^{1/2}. \tag{16}$$

It is therefore seen that for any choice of α and b one may make u_1 as large as one chooses by taking ζ large enough. A tabulation of C for $b=0.5$ and various values of α is given in table 1.

TABLE 1

α	C for $b=0.5$	τ	τ^{-1}	$0.066\tau^{-1}$	u_3
0.02	24.96	$6.34\times10^{-4}\zeta^{-1}$	1576ζ	104ζ	2ζ
0.08	6.09	$1.02\times10^{-2}\zeta^{-1}$	98.5ζ	6.50ζ	2ζ
0.30	1.19	$0.143\zeta^{-1}$	7.00ζ	0.462ζ	0.462ζ
0.60	0.34	$.571\zeta^{-1}$	1.75ζ	$.116\zeta$	$.116\zeta$
1.20	.09	$2.28\zeta^{-1}$	0.44ζ	$.029\zeta$	$.029\zeta$

From (11) and (13)

$$T=\frac{2j}{\zeta}\int_0^{u_1}\frac{(1+\epsilon_1)e^{-u}du}{\eta^2\left[1+\left(\frac{2}{\pi}\log\frac{\Gamma\eta\alpha}{2}\right)^2\right]}+\frac{2j}{\zeta}\int_{u_1}^{\infty}\frac{e^{-u}}{\eta^2 D(\eta\alpha)}du. \tag{17}$$

Since it is possible to choose an A such that $|D(\eta\alpha)|>(A\alpha|\eta|)^{-1}$ for all $|\eta\alpha|>b$, we may make use of (12) to obtain the inequality

$$\left|\frac{2j}{\zeta}\int_{u_1}^{\infty}\frac{e^{-u}du}{\eta^2 D(\eta\alpha)}\right|\leq\frac{2A\alpha}{\zeta}\int_{u_1}^{\infty}\frac{e^{-u}}{|\eta|}du\leq A\alpha\sqrt{\frac{2}{u_1\zeta}}\,e^{-u_1} \tag{18}$$

which, using (15), can be written

$$\left|\frac{2j}{\zeta}\int_{u_1}^{\infty}\frac{e^{-u}du}{\eta^2 D(\eta\alpha)}\right|\leq\frac{A\alpha}{\zeta}\sqrt{\frac{2}{C}}\,e^{-C\zeta}. \tag{19}$$

Thus with the neglect of ϵ_1 in (17) and exponential term in ζ we may write

$$T\cong\frac{2j}{\zeta}\int_0^{M}\frac{e^{-u}du}{\eta^2\left[1+\left(\frac{2}{\pi}\log\frac{\Gamma\eta\alpha}{2}\right)^2\right]} \tag{20}$$

where M is u_1 or any quantity very large compared to 1. Since the integrand of the first integral in (17) has nearly a constant angle, the percentage error in dropping the integral involving ϵ_1 is certainly not much more than the maximum error in dropping ϵ_1 in (13).

Expansion in (20) of $[(u/\zeta)+2j]^{-1}$ by the binomial theorem leads to a series in reciprocal powers of ζ

$$T\cong\sum_{n=0}^{\infty}\left(\frac{j}{2\zeta}\right)^n T_n, \tag{21}$$

where

$$T_n=\int_0^{M}\frac{u^{n-1}e^{-u}du}{1+\left(\frac{1}{\pi}\log\frac{\Gamma^2\eta^2\alpha^2}{4}\right)^2}. \tag{22}$$

At this point to insure uniform convergence of the binomial series, M is to be taken slightly smaller than u_2, where u_2 is the smaller of u_1 and 2ζ.

From (12)

$$\eta^2=\eta_0^2-\frac{1}{4}\eta_0^4 \tag{23}$$

where

$$\eta_0=\sqrt{\frac{2ju}{\zeta}}. \tag{24}$$

Consider now a function f of η^2 in the range in which $u<<\zeta$. By a Taylor expansion

$$f(\eta^2)\cong f(\eta_0^2)-\frac{1}{4}\eta_0^4\frac{df(\eta_0^2)}{d\eta_0^2}. \tag{25}$$

Since, however,

$$\frac{df}{d\eta_0^2}=\frac{du}{d\eta_0^2}\frac{df}{du}=-\frac{j\zeta}{2}\frac{df}{du}$$

(25) may be written

$$f(\eta^2)\cong f(\eta_0^2)-\frac{ju^2}{2\zeta}\frac{df(\eta_0^2)}{du}. \tag{26}$$

Applying (26) to (22), we have

$$T_n \cong \int_0^M \frac{u^{n-1}e^{-u}du}{1+\left(\frac{1}{\pi}\log\frac{\Gamma^2\eta_0^2\alpha^2}{4}\right)^2} - \frac{j}{2\zeta}\int_0^M u^{n+1}e^{-u}\frac{d}{du}\left[\frac{1}{1+\left(\frac{1}{\pi}\log\frac{\Gamma^2\eta_0^2\alpha^2}{4}\right)^2}\right]du.$$

The application of (26) in the outer part of the range $0 \leq u \leq M$ will not be very accurate, but since the integrand drops off as e^{-u}, one may ignore this. Integrating the last integral by parts and neglecting the term containing e^{-M} in the integrated part, we have

$$T_n \cong T_n^0 + \frac{j}{2\zeta}[(n+1)T_{n+1}^0 - T_{n+2}^0] \tag{27}$$

where the superscript zero indicates that η is to be replaced by η_0.

If in (21) one neglects terms of $(1|\zeta)^2$ and higher, then one may write that

$$T \cong T_0 + \frac{j}{2\zeta}T_1.$$

By (27) this becomes

$$T \cong T_0^0 + \frac{j}{2\zeta}[2T_1^0 - T_2^0]. \tag{28}$$

Equations (10) and (28) provide us with an asymptotic expression for $I(z)$ involving integrals of the form

$$T_n^0 = \int_0^M \frac{u^{n-1}e^{-u}du}{1+\left(\frac{1}{\pi}\log\frac{\Gamma^2\eta_0^2\alpha^2}{4}\right)^2}$$
$$= \int_0^M \frac{u^{n-1}e^{-u}du}{1+\left(\frac{1}{\pi}\log u\tau j\right)^2} \tag{29}$$

where use has been made of (24) and where

$$\tau = \frac{\Gamma^2\alpha^2}{2\zeta}. \tag{30}$$

4. Asymptotic Series for $I(z)$

Integrating (29) by parts using the function

$$A(x) = \frac{\pi}{2} + \arctan x \tag{31}$$

we have

$$\frac{1}{\pi}T_n^0 = \left[u^n e^{-u}A\left(\frac{1}{\pi}\log u\tau j\right)\right]_0^M + \int_0^M (u^n - nu^{n-1})e^{-u}A\left(\frac{1}{\pi}\log u\tau j\right)du.$$

Since $A\left(\frac{1}{\pi}\log u\tau j\right) \rightarrow 0$ as $u \rightarrow 0$ and M is large, this reduces to

$$T_n^0 = W_n - nW_{n-1} + M^n e^{-M}A\left(\frac{1}{\pi}\log M\tau j\right)$$
$$\cong W_n - nW_{n-1} \tag{32}$$

where

$$W_n = \pi\int_0^M u^n e^{-u}A\left(\frac{1}{\pi}\log u\tau j\right)du. \tag{33}$$

Collecting our results thus far, we have from (10), (28), and (32) that

$$I(z)\cong\frac{2V}{\pi}\frac{e^{-j\zeta}}{Z_0}\left[W_0+\frac{j}{2\zeta}(4W_1-2W_0-W_2)\right]. \tag{34}$$

Corresponding to the expansion of the arctangent in reciprocal powers of the argument, one has

$$A\left(\frac{1}{\pi}\log u\tau j\right)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\left(\frac{1}{\pi}\log u\tau j\right)^{-(2k+1)}. \tag{35}$$

This series converges for

$$\left|\frac{1}{\pi}\log u\tau j\right|>1$$

and thus will converge when

$$-\log u\tau>\frac{\sqrt{3}}{2}\pi\text{ or }u<0.066\tau^{-1}. \tag{36}$$

Let u_3 be the smallest of the quantities $u_1=C\zeta$, 2ζ, and $0.066\tau^{-1}$ then (35) converges uniformly for $0\leq u\leq M<u_3$, and with neglect of terms of order e^{-M}

$$W_n\simeq\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\pi^{2k+2}I_{nk} \tag{37}$$

with

$$I_{nk}=-\int_0^M\frac{u^ne^{-u}du}{(\log u\tau j)^{2k+1}}. \tag{38}$$

This follows from the fact that a uniformly convergent series may be integrated term by term. It is here understood that M is to be taken only negligibly smaller than u_3. It is seen from table 1 that u_3, and hence M, can be taken as large as one desires by increasing ζ.

Equation (38) may be written

$$I_{nk}=\frac{-1}{(\log\tau j)^{2k+1}}\int_0^M\frac{u^ne^{-u}du}{\left[1+\frac{\log u}{\log\tau j}\right]^{2k+1}}. \tag{39}$$

Using Taylor's series with a remainder term to expand the denominator, we have (see Widder [1961])

$$\left[1+\frac{\log u}{\log\tau j}\right]^{-(2k+1)}=\sum_{s=0}^{N}(2k+1)^{\{s\}}\left(\frac{-\log u}{\log\tau j}\right)^s+R_N \tag{40}$$

where $\{s\}$ as a superscript is a pseudoexponent defined by the equation

$$(2k+1)^{\{s\}}=\begin{cases}(2k+1)(2k+2)\ldots(2k+s), & s>0\\ 1, & s=0\\ [(2k)(2k-1)\ldots(2k+s+1)]^{-1}, & s<0\end{cases} \tag{41}$$

and

$$R_N=\frac{(2k+1)^{\{N+1\}}}{N!}\int_0^{\frac{-\log u}{\log\tau j}}\frac{\left[\frac{-\log u}{\log\tau j}-t\right]^N}{(1-t)^{2k+N+2}}dt. \tag{42}$$

Letting $t=-\log v/\log\tau j$ in (42), one may write

$$R_N=-\frac{(2k+1)^{\{N+1\}}}{N!(\log\tau j)^{N+1}}\int_1^u\frac{\left(\log\frac{v}{u}\right)^N}{\left(\frac{\log v\tau j}{\log\tau j}\right)^{2k+N+2}}\frac{dv}{v}. \tag{43}$$

On substituting from (40) into (39), one obtains the equation

$$I_{nk}=\frac{-1}{(\log \tau j)^{2k+1}}\left[\sum_{s=0}^{N}\frac{(2k+1)^{\{s\}}}{s!}\,\frac{C_{ns}}{(\log \tau j)^{s}}+R'_N(n)\right] \tag{44}$$

where

$$C_{ns}=(-1)^s\int_0^M u^n e^{-u}(\log u)^s du \tag{45}$$

and

$$R'_N(n)=\int_0^M u^n e^{-u}R_N du. \tag{46}$$

It will be shown later in section 5 that as $\zeta\rightarrow\infty$ and hence $\tau\rightarrow 0$ the remainder term $R'_N(n)$ approaches zero as $|\log \tau j|^{-(N+1)}$. This insures that (44) is a legitimate asymptotic expansion of I_{nk}.

It is convenient to replace (45) by the equation

$$C_{ns}=(-1)^s\int_0^\infty u^n e^{-u}(\log u)^s du \tag{47}$$

which involves again only the neglect of a term proportional to e^{-M} and to write (44) in the form

$$I_{nk}\cong\frac{-1}{(\log \tau j)^{2k+1}}\sum_{s=0}^{(\infty)}\frac{(2k+1)^{\{s\}}}{s!}\,\frac{C_{ns}}{(\log \tau j)^{s}} \tag{48}$$

to indicate an asymptotic series that must be terminated at some finite but unspecified term. The remainder term is then very roughly indicated by the first neglected term. A precise determination of the overall error is made later in section 7 by comparing the results obtained from our asymptotic series with those obtained numerically by Duncan [1962].

Integrating (47) by parts, we have

$$(-1)^s C_{ns}=\left[u^{n+1}e^{-u}\,\frac{(\log u)^{s+1}}{s+1}\right]_0^\infty+\frac{1}{s+1}\int_0^\infty [u^{n+1}-(n+1)u^n]e^{-u}(\log u)^{s+1}du$$

or

$$C_{n+1,\,s+1}=(n+1)C_{n,\,s+1}-(s+1)C_{n,\,s}.$$

Thus

$$C_{n,\,s}=nC_{n-1,\,s}-sC_{n-1,\,s-1} \tag{49}$$

and, in particular,

$$C_{1s}=C_{0s}-sC_{0,\,s-1} \tag{50}$$

$$\begin{aligned}C_{2s}&=2C_{1s}-sC_{1,\,s-1}\\&=2C_{0s}-3sC_{0,\,s-1}+s(s-1)C_{0,\,s-2}.\end{aligned} \tag{51}$$

From (37) and (48), since in (37) W_n can be approximated by a truncated series,

$$\begin{aligned}&W_0+\frac{j}{2\zeta}[4W_1-2W_0-W_2]\\&\quad=\pi\sum_{k=0}^{(\infty)}\frac{(-1)^{k+1}}{2k+1}\left(\frac{\pi}{\log \tau j}\right)^{2k+1}\sum_{s=0}^{(\infty)}\frac{(2k+1)^{\{s\}}}{s!}\,\frac{1}{(\log \tau j)^{s}}\left[C_{0s}+\frac{j}{2\zeta}(4C_{1s}-2C_{0s}-C_{2s})\right]\end{aligned} \tag{52}$$

and thus, finally, by (34), (41), (50), and (51)

$$I(z)\cong\frac{2\pi Ve^{-j\zeta}}{Z_0}\sum_{k=0}^{(\infty)}\sum_{s=0}^{(\infty)}\frac{(-1)^{k+1}(2k+2)^{\{s-1\}}\pi^{2k}}{s!(\log \tau j)^{2k+s+1}}\left[C_s-\frac{j}{2\zeta}(sC_{s-1}+s(s-1)C_{s-2})\right] \tag{53}$$

where

$$C_s\equiv C_{0s}=(-1)^s\int_0^\infty e^{-u}(\log u)^s du. \tag{54}$$

Equation (53) may be written

$$I(z)\cong\frac{2\pi Ve^{-j\zeta}}{Z_0}\left[\sum_{n=1}^{(\infty)}\frac{A_n}{(\log p\zeta)^n}+\frac{j}{\zeta}\sum_{n=2}^{(\infty)}\frac{B_n}{(\log p\zeta)^n}\right] \tag{55}$$

where A's and B's are constants and $-\log \tau j$ is replaced by $\log p\zeta$; thus

$$p=\frac{-2j}{\Gamma^2\alpha^2}. \tag{56}$$

These are the same equations as (2) and (3) in the introduction. Vainshtein [1959] has obtained a similar type of expansion for a related integral.

The second series in (55) is not, strictly speaking, a part of an asymptotic expansion of $I(z)$ since terms in $1/\zeta$ fall off faster than any power of $(\log p\zeta)^{-1}$. It can, however, be used to increase the range of ζ over which (55) is a valid approximation.

By direct comparison of (53) and (55) we see that

$$\begin{aligned}
&A_1=C_0 && \\
&A_2=-C_1 && B_2=\frac{1}{2}C_0 \\
&A_3=-\frac{\pi^2}{3}C_0+C_2 && B_3=-(C_0+C_1) \\
&A_4=\pi^2C_1-C_3 && B_4=-\frac{\pi^2}{2}C_0+3C_1+\frac{3}{2}C_2 \\
&A_5=\frac{\pi^4}{5}C_0-2\pi^2C_2+C_4 && B_5=2\pi^2(C_0+C_1)-6C_2-2C_3
\end{aligned} \tag{57}$$

5. Magnitude of the Remainder Term $R'_N(n)$

If u is in the range $0\leq u\leq \tau^{-1/2}$, then from (43)

$$|R_N|<\frac{(2k+1)^{\{N+1\}}}{N!\,|\log \tau j|^{N+1}}\left|\frac{\log\tau+\frac{\pi}{2}j}{\frac{1}{2}\log\tau+\frac{\pi}{2}j}\right|^{2k+N+2}\left|\left[\frac{\left(\log\frac{v}{u}\right)^{N+1}}{N+1}\right]_1^u\right|<\frac{(2k+1)^{\{N+1\}}2^{2k+N+2}\,|\log u|^{N+1}}{(N+1)!\,|\log \tau j|^{N+1}}. \tag{58}$$

Since $|\log v\tau j|$ in (43) for real v is always greater than $\pi/2$, we have

$$|R_N|<\frac{(2k+1)^{\{N+1\}}|\log \tau j|^{2k+1}|\log u|^{N+1}}{(N+1)!\left(\frac{\pi}{2}\right)^{2k+N+2}} \tag{59}$$

even outside the above range.

These inequalities applied to (46) yield an upper bound for $R'_N(n)$ expressed by

$$|R'_N(n)|<\frac{(2k+1)^{\{N+1\}}}{(N+1)!}\left\{\frac{2^{2k+N+2}}{|\log \tau j|^{N+1}}\int_0^{\tau^{-1/2}}u^ne^{-u}|\log u|^{N+1}du\right.$$
$$\left.+\frac{|\log \tau j|^{2k+1}}{\left(\frac{\pi}{2}\right)^{2k+N+2}}\int_{\tau^{-1/2}}^{M}u^ne^{-u}|\log u|^{N+1}du\right\}. \tag{60}$$

The second integral approaches zero as $e^{-\tau^{-1/2}}$ and thus for small τ (large ζ) is negligible. One sees therefore that $R'_N(n)$ goes to zero as

$$\frac{\text{constant}}{|\log \tau j|^{N+1}}. \tag{61}$$

6. Computation of the C_n

From (54)

$$C_n=(-1)^n\int_0^\infty e^{-u}(\log u)^n du=P_n+(-1)^n Q_n \tag{62}$$

where

$$P_n=(-1)^n\int_0^1 e^{-u}(\log u)^n du>0 \tag{63}$$

and

$$Q_n=\int_1^\infty e^{-u}(\log u)^n du>0. \tag{64}$$

In (63) let $x=-\log u$; then

$$\begin{aligned} P_n&=\int_0^\infty e^{-\exp(-x)}x^n e^{-x}dx \\ &=\sum_{p=0}^\infty \frac{(-1)^p}{p!}\int_0^\infty x^n e^{-(p+1)x}dx=n!\sum_{p=0}^\infty \frac{(-1)^p}{p!(p+1)^{n+1}} \end{aligned} \tag{65}$$

and one obtains a convenient alternating series for P_n.

To obtain the Q_n we set

$$Q_n=\int_1^M e^{-u}(\log u)^n du+e_n(M) \tag{66}$$

where the truncation error $e_n(M)$ in using M rather than infinity as the upper limit is given by

$$e_n(M)=\int_M^\infty e^{-u}(\log u)^n du=e^{-M}\int_0^\infty e^{-x}\left[\log M+\log\left(1+\frac{x}{M}\right)\right]^n dx. \tag{67}$$

Now $\log\left(1+\frac{x}{M}\right)\leq\frac{x}{M}$ for all $x\geq 0$ and hence $e_n(M)$ is less than

$$E_n(M)\equiv\frac{e^{-M}}{M^n}\int_0^\infty e^{-x}(M\log M+x)^n dx$$

which on successive integration by parts reduces to

$$\begin{aligned} E_n(M)&=\frac{e^{-M}}{M^n}n!\sum_{k=0}^n \frac{(M\log M)^k}{k!} \\ &=(\log M)^n e^{-M}\left[1+\frac{n}{M\log M}+\frac{n(n-1)}{(M\log M)^2}+\ldots+\frac{n!}{(M\log M)^n}\right]. \end{aligned} \tag{68}$$

Setting $n=5$ and $M=16$, we have

$$E_5(16)=(2.77)^5 e^{-16}[1.124]=2.08\times 10^{-5}.$$

For smaller n the error is even less. Thus, we may with negligible error set

$$Q_n=\int_1^{16} e^{-u}(\log u)^n du,\qquad n=0,1,2,\ldots \tag{69}$$

and evaluate these integrals by numerical quadrature. Actually, one knows that

$$C_0=\int_0^\infty e^{-u}du=1 \tag{70}$$

and from standard integral tables that

$$C_1=\int_0^\infty e^{-u}\log u\,du=\gamma\equiv\text{Euler's constant}. \tag{71}$$

Hence numerical integration is not required in (69) for $n=0$ and $n=1$.

A table of the C_n and the A_n and B_n obtained from them by (57) is given below:

TABLE 2

n	C_n	A_n	B_n
0	1.0000		
1	0.5772	1.0000	
2	1.9781	−0.5772	0.5000
3	5.4449	−1.3118	−1.5772
4	23.561	0.2520	−0.2360
5	117.84	3.9969	8.3743

7. Accuracy of Results Obtained Using Asymptotic Series

Since several approximations were made in deriving the asymptotic series (55) for the current on an infinite cylindrical antenna, it is important to check the results against the very accurate results obtained by Duncan [1962] by numerical integration. For this purpose only the first five terms of the A coefficient series were used. The computations were made for rather thick antennas, $\alpha=0.02$, 0.08, 0.30, 0.60, and 1.20, and some of the results are shown in figures 2 to 10.

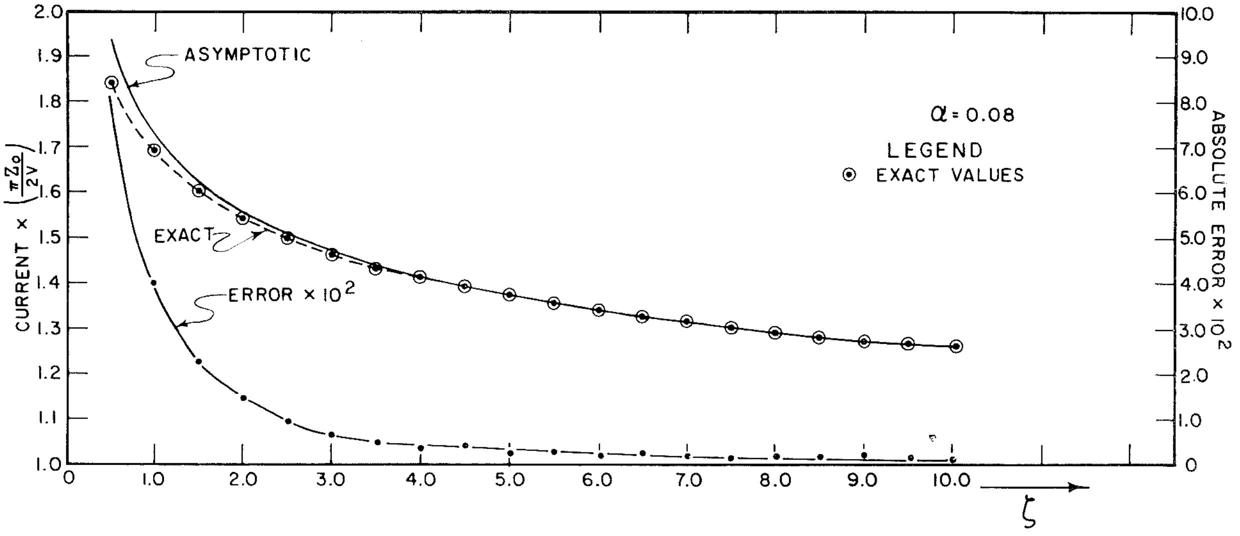

FIGURE 2. *Magnitude of the current for $\alpha=0.08$.*
"Asymptotic" refers to first five terms of the A series in (55).

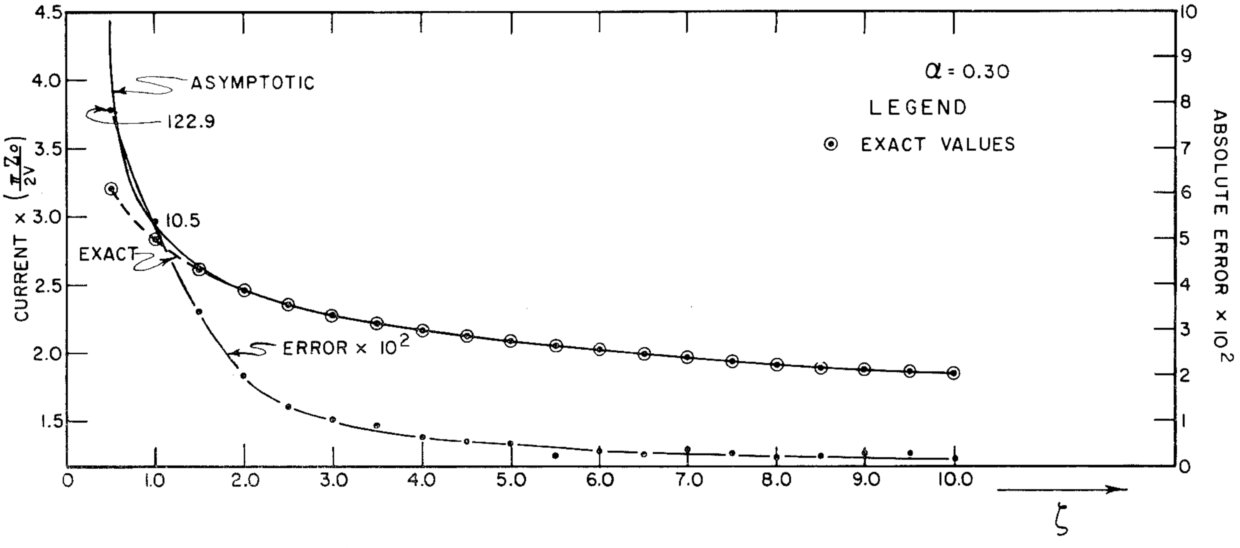

FIGURE 3. *Magnitude of the current for $\alpha=0.30$.*
"Asymptotic" refers to first five terms of the A series in (55).

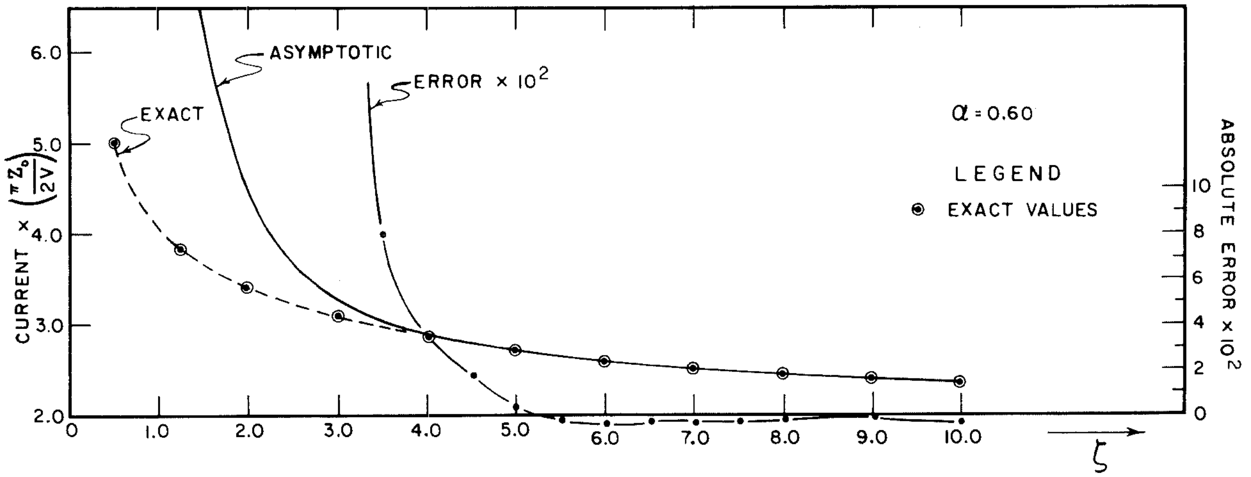

Figure 4. *Magnitude of the current for α=0.60.*
"Asymptotic" refers to first five terms of the A series in (55).

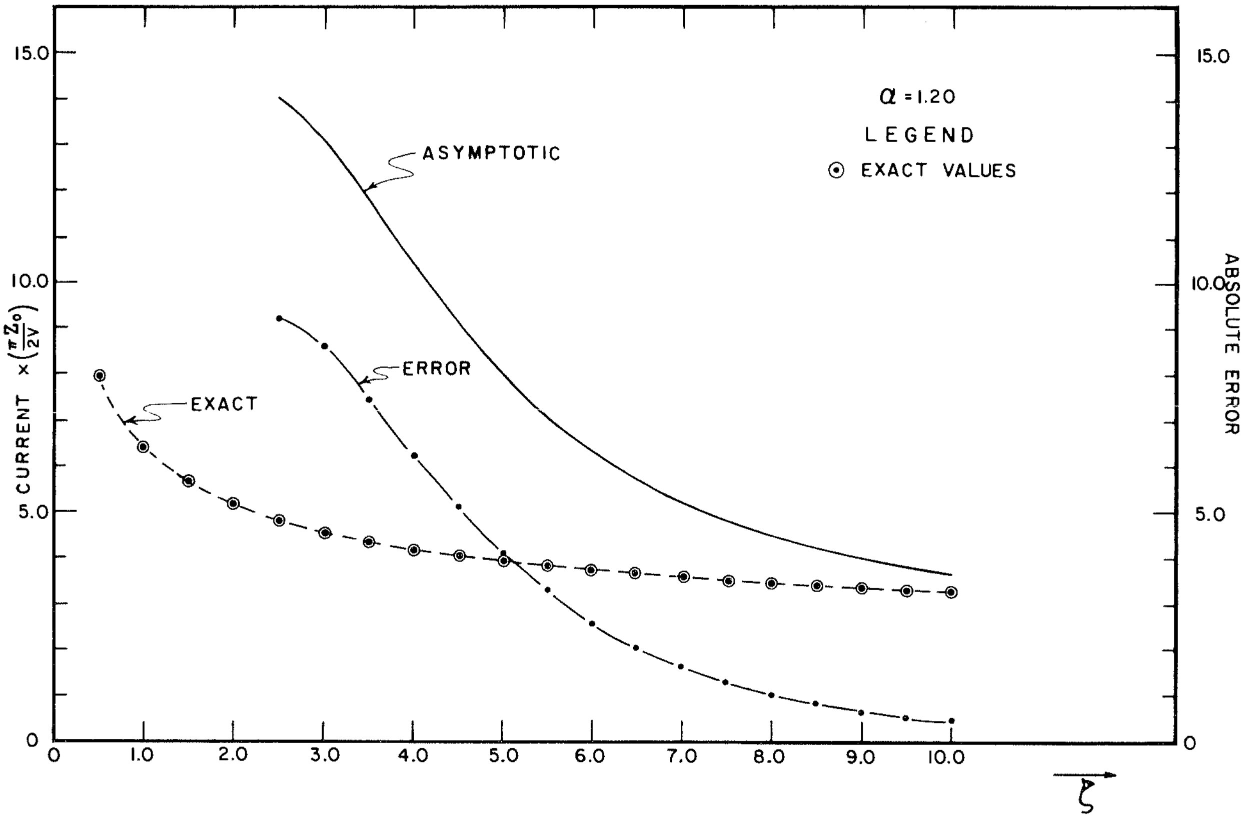

Figure 5. *Magnitude of the current for α=1.20.*
"Asymptotic" refers to first five terms of the A series in (55).

The results for α=0.02 are not given since at α=0.08 (see figs. 2, 6, and 9) the results are already quite accurate even as close as a quarter wavelength from the gap. For thinner antennas the agreement would be even better. One observes (see figs. 4, 7, and 10) that for α=0.60 the asymptotic series begins to give accurate results only when ζ is greater than 3.5 or 4. For a circumference approaching one wavelength (α=1) the asymptotic expansion ceases to be even approximately valid for determining the magnitude of the current as can be seen from figure 5 (α=1.2). Surprisingly enough figure 8 shows that the phase angle of the current for α=1.2 is somewhat better predicted, being less than 20 deg off from the true value.

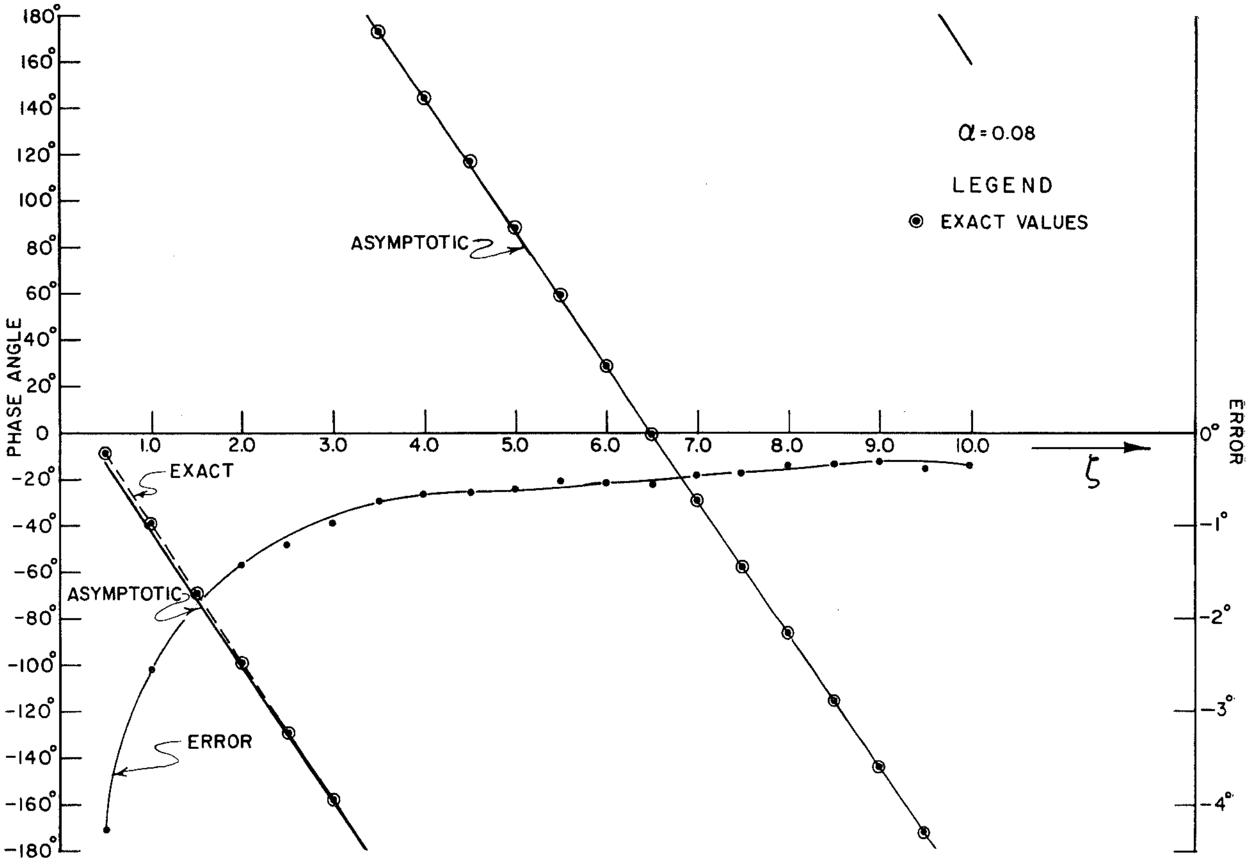

FIGURE 6. *Phase angle of the current for α=0.08.*
"Asymptotic" refers to first five terms of the A series in (55).

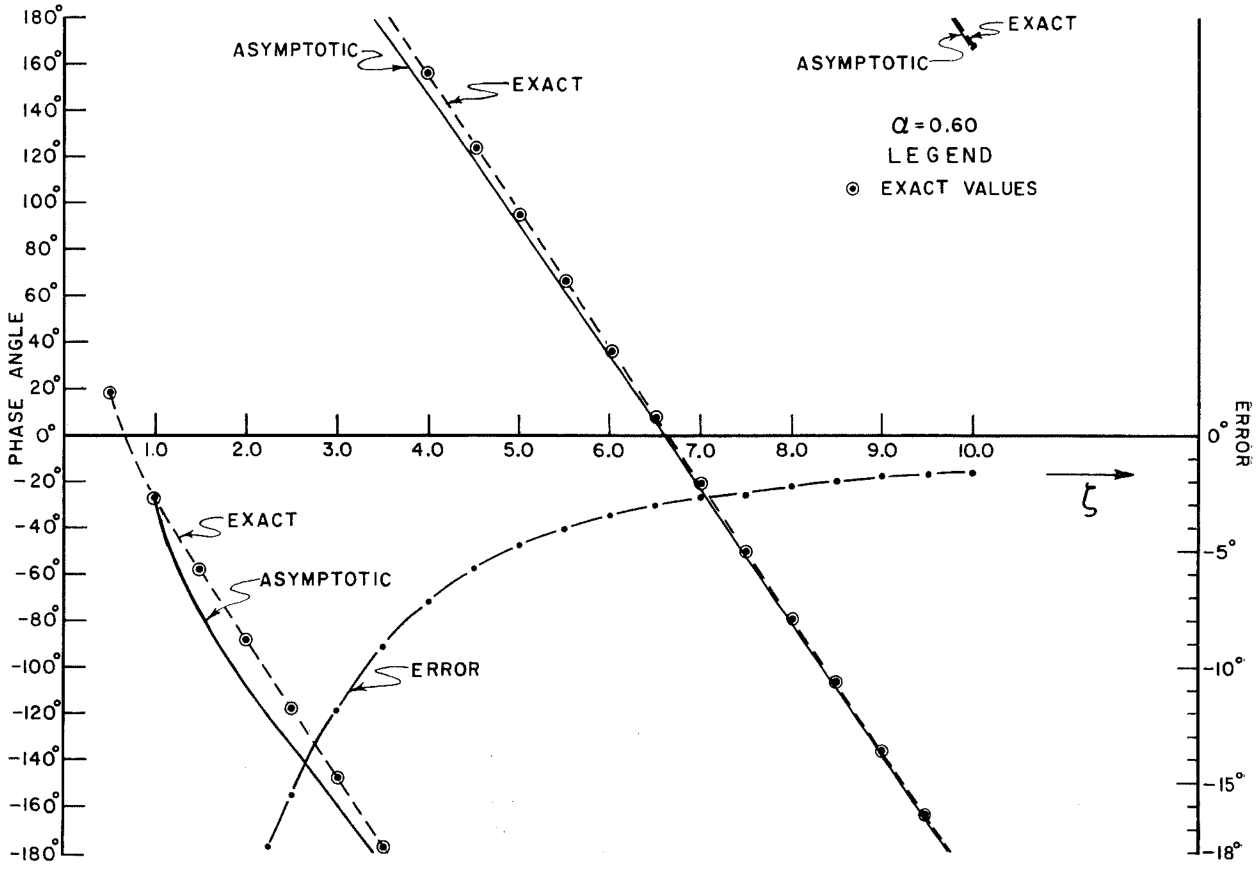

FIGURE 7. *Phase angle of the current for α=0.60.*
"Asymptotic" refers to first five terms of the A series in (55).

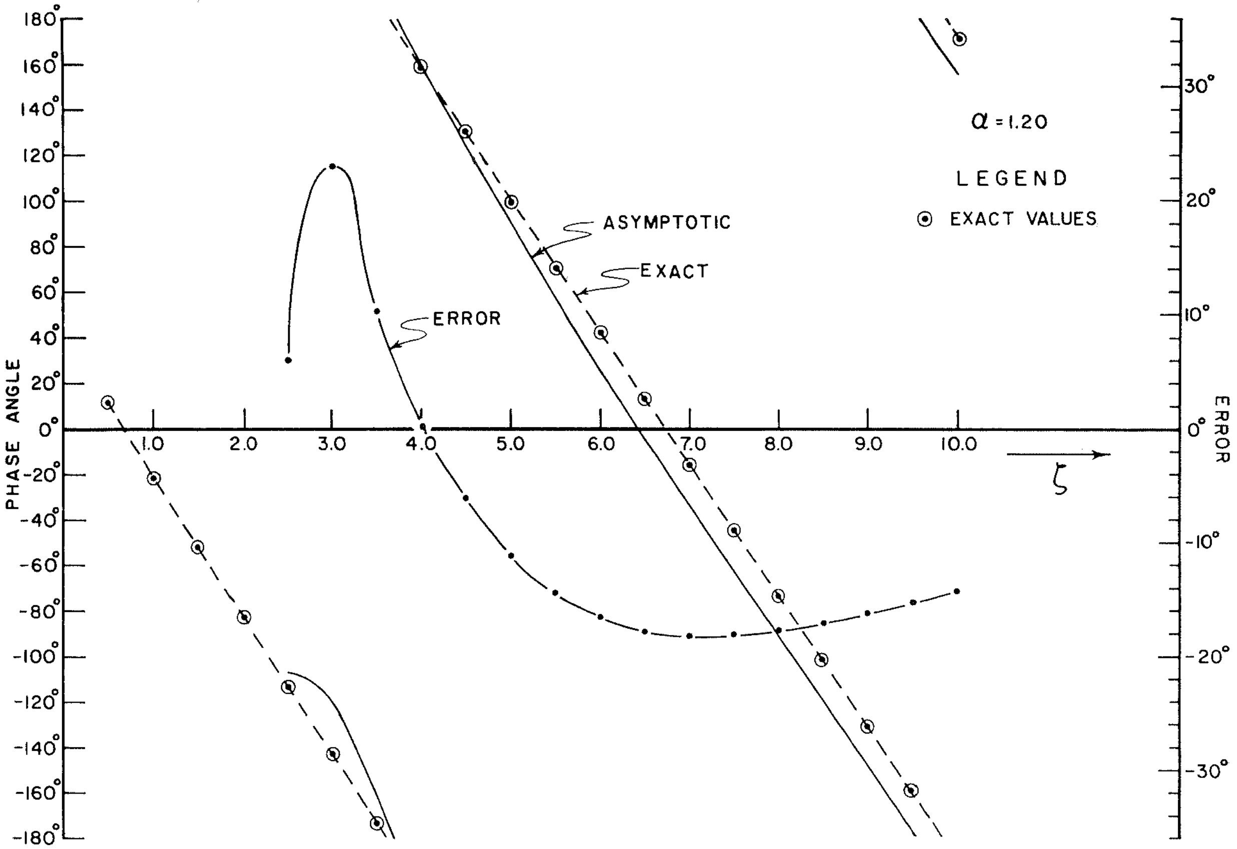

FIGURE 8. *Phase angle of the current for α=1.20.*

"Asymptotic" refers to first five terms of the A series in (55).

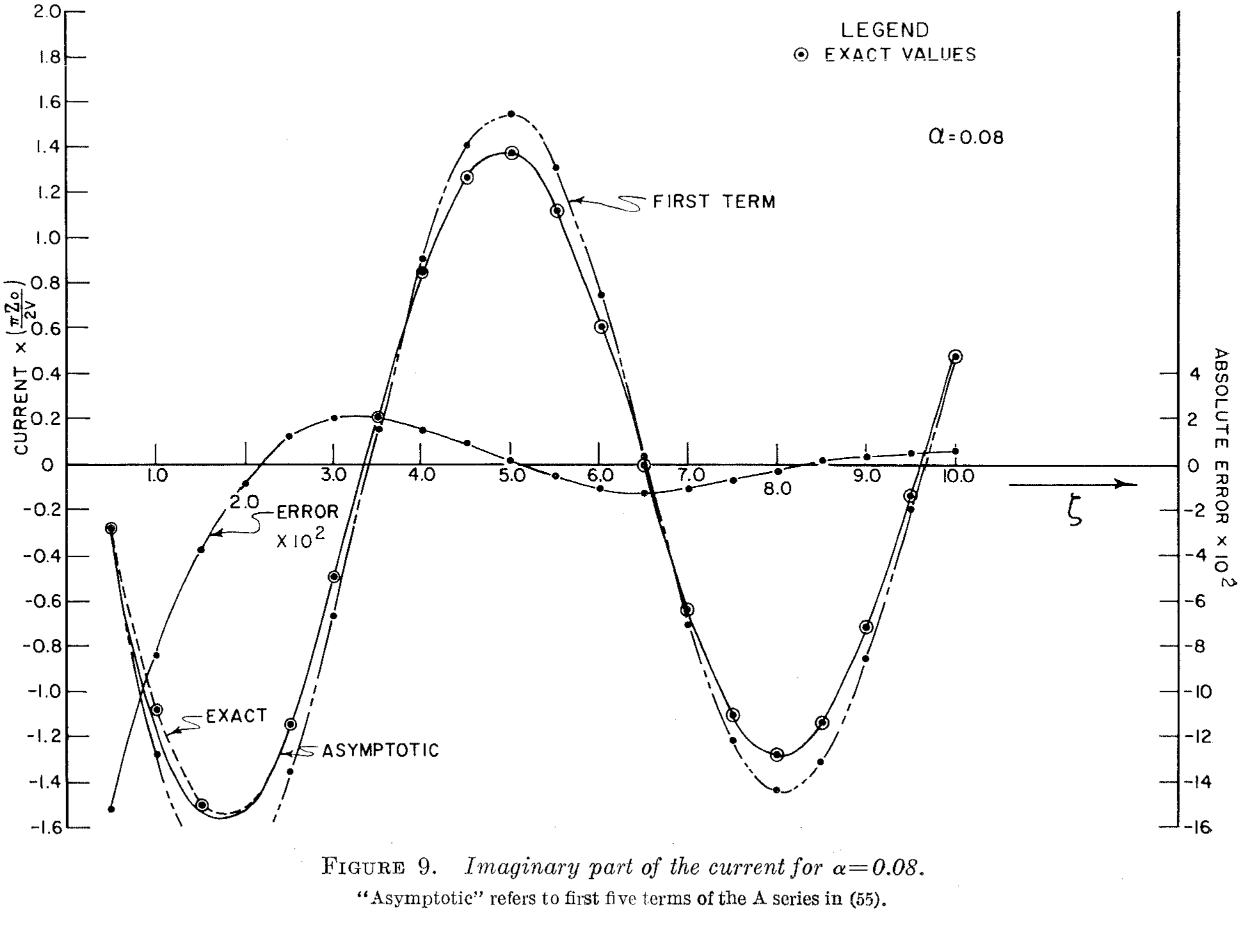

FIGURE 9. *Imaginary part of the current for α=0.08.*

"Asymptotic" refers to first five terms of the A series in (55).

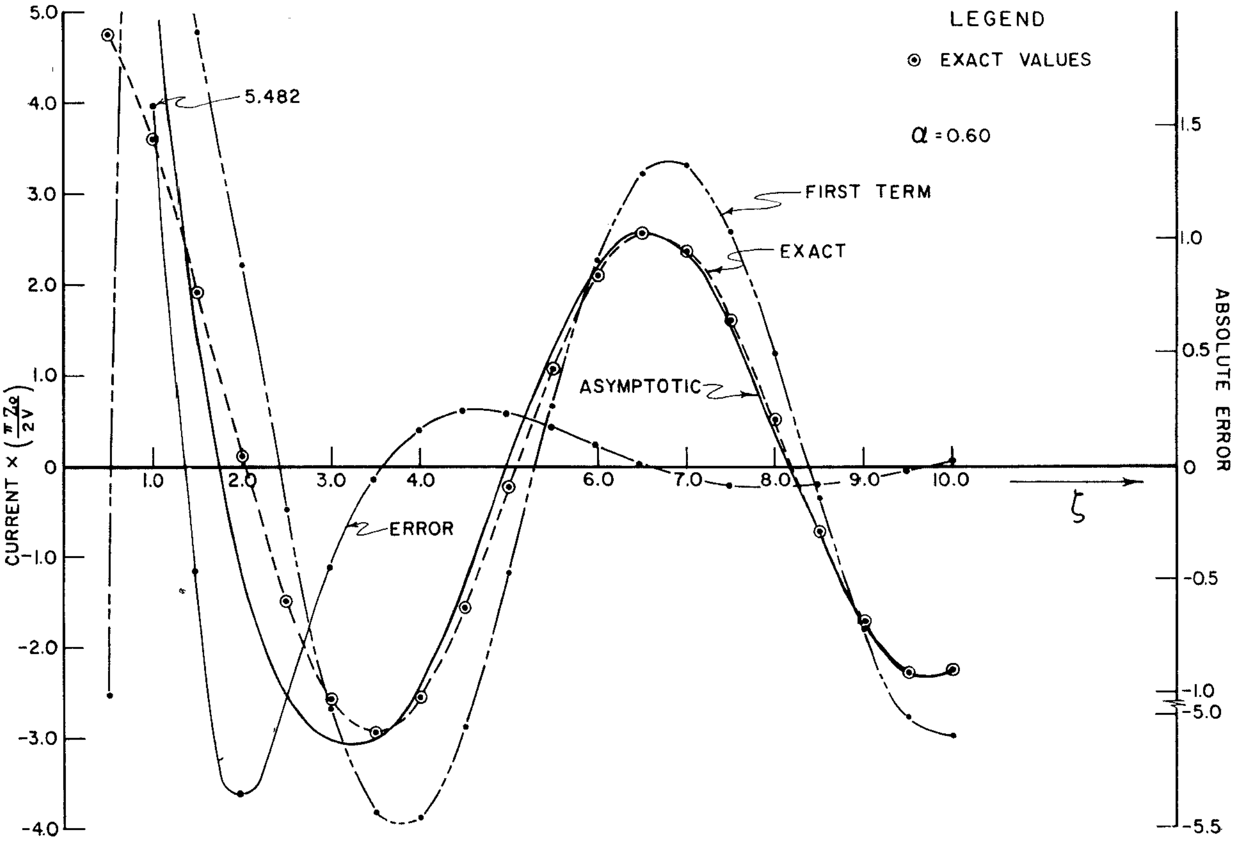

FIGURE 10. *Real part of the current for α=0.60.*

"Asymptotic" refers to first five terms of the A series in (55).

The first term of the asymptotic expansion (55) taken alone yields the asymptotic formula

$$I(z) \cong \frac{2\pi V}{Z_0} \frac{e^{-jk|z|}}{\log \dfrac{-2j|z|}{\Gamma^2 ka^2}} \tag{72}$$

which is accurate for antennas of the thickness here considered only for rather large $\zeta = k|z|$. This is shown in figures 9 and 10. Even for the $\alpha = 0.08$ case the plot of the first term fails to give close agreement even at a ζ of the order of 10. These two figures illustrate, in addition, the degree to which the first five terms give the real and imaginary parts of the current. Since the accuracy is approximately the same for each part, we have given only the imaginary part for $\alpha = 0.08$ and only the real part for $\alpha = 0.60$. In the latter, of course, the agreement is not too good.

Formulas similar to (72) have been obtained by others. Chen and Keller [1962] obtain the same formula but with the argument of the log the negative of that in (72). Northover [1958] derives a formula in which the argument of the log is $z/2ka^2$. Hallén's [1956] formula agrees with (72) except that Γ is to the first power. All of these formulas are correct in that they lead to the right functional behavior for $\zeta \to \infty$. It has been checked, on the other hand, that in the case $\alpha = 0.08$, at least, formula (72) is more accurate for finite ζ than any of these variations of it.

We have also determined that for $\alpha = 0.08$ and 0.60 the error remaining after taking the first five terms of the A coefficient series is given for intermediate ζ very nearly by the first two or three terms of the B coefficient series of (55).

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